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WEAK FLATNESS CRITERIA FOR CODIMENSION 2 SPHERES IN CODIMENSION 1 MANIFOLDS *

R.J. DAVERMAN

Department of Mathematics, University of Tennessee, Knoxville, Tenn. 37916, U.S.A.

T.B. RUSHING

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, U.S.A.

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The "Kiec problem" is to determine which submanifolds of S^n have the property that all their subspheres and subcells are flat. In this paper we study the related problem of determining submanifolds of S^n having all subspheres weakly flat and all subcells cellular.

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weakly flat — cell-like
1-ALG — cellular
cellularity criterion

1. Introduction

A k-sphere K in S^n (resp. E^n) is said to be *weakly flat* if $S^n - K$ (resp. $E^n - K$) is homeomorphic to the complement of the standard k-sphere in S^n (resp. E^n). The main purpose of this paper is to determine $(n-1)$-manifolds in S^n in which each k-sphere, $k \leqslant n-2$, is weakly flat. In particular, we establish the following results:

each k-sphere ($k \geqslant 2$) in the boundary of a cellular m-cell in S^n ($n \geqslant 5$) is weakly flat (Theorem 3.9);

each sphere in a factored $(n-1)$-string in E^n is weakly flat (Corollary 4.7);

for an $(n-1)$-sphere S in S^n ($n \geqslant 4$), each sphere in $\Sigma(S) \subset \Sigma(S^n) = S^{n+1}$ (the natural suspension of S in S^{n+1}) that contains at most one of the suspension points is weakly flat (Corollary 4.9);

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for an $(n-1)$-sphere S in S^n $(n \geqslant 4)$ having simply connected complementary domains, each sphere in $\Sigma(S) \subset S^{n+1}$ is weakly flat (Theorem 4.5).

In each of the above results the conclusion "each sphere is weakly flat" may be replaced by "each cell-like subset is cellular".

The fundamental definition given below pertains to the following situation: S is an $(n-1)$-manifold embedded as a closed separating subset of an n-manifold Q, and W is a component of $Q - S$. A subset X of S is said to be *unsnarled from* W at a point $x \in S$ if, for each open subset U of Q containing X, there exists a neighborhood N_x of x such that each loop in $N_x \cap W$ is contractible in $U - X$. In addition, X is said to be *unsnarled from* W if it is unsnarled from W at each point of X.

In order to depict the importance of the unsnarled property, we mention two results, the first of which gives the equivalence between the unsnarled property and weak flatness of subspheres, and the second furnishes a form of the property that is preserved by suspensions. First, for an $(n-1)$-sphere S in S^n $(n \neq 4)$ having complementary domains W_1 and W_2, a k-sphere K in S is weakly flat iff K is unsnarled from both W_1 and W_2 (Theorem 3.7). Second, for an $(n-1)$-sphere S in S^n such that each nowhere dense compactum $X \subset S$ is unsnarled from a component W of $S^n - S$, each nowhere dense compactum in $S' = \Sigma(S)$ is unsnarled from the component of $\Sigma(S^n) - S'$ containing W (Lemma 4.2). Whenever the above holds in both complementary domains and $n + 1 \geqslant 5$, it then follows (Theorem 4.3) that each sphere in S' is cellular.

The cellularity of cell-like subsets of S^n $(n \neq 4)$ is readily determined, of course, in terms of McMillan's Cellularity Criterion [19]. Similarly, the weak flatness of a sphere in S^n $(n \neq 4)$ also can be determined in a homotopy theoretic fashion.

Weak flatness criterion. *A k-sphere K in S^n, $n \geqslant 5$ (such that $S^n - K$ has the homotopy type of S^1 in case $k = n - 2$) is weakly flat iff Σ is globally 1-alg in S^n.*

This criterion combines work of McMillan [19, Theorem 4] for $k = n - 1$, Rushing and Hollingsworth [17, Theorem 1] for $k = n - 2$, Duvall [13, Theorem 2.1] for $2 \leqslant k \leqslant n - 3$, and Daverman [8, Theorem 1] for $k = 1$. As in [8] and [17], a subset X of an n-manifold Q^n is *globally* 1-*alg* if for each open set U containing X there exists an open set V containing X, $V \subset U$, such that each loop in $V - X$ that is null-homologous in $V - X$ is null-homotopic in $U - X$. In case X denotes either a k-sphere ($k \neq 1, n-2$) in Q^n or a cell-like subset of Q^n ($n \geqslant 3$), X is globally 1-alg iff X satisfies the Cellularity Criterion.

This paper was conceived upon the observation that techniques of [9] would detect the globally 1-alg property for codimension 2 subspheres of certain suspended codimension 1 spheres and that the complement of these codimension 2 subspheres would have the homotopy type of S^1; as a result, one has access to many examples for which the Weak Flatness Criterion applies. It is especially interesting to witness the natural appearance of many codimension 2 spheres having complements homotopically equivalent to S^1, thus allowing application of [17] to determine weak flatness, in view of the fact that at present there is no homotopy-theoretic characterization of those codimension 2 spheres in S^n ($n \geqslant 4$) whose complements are homeomorphic to the complement of a locally flat but knotted codimension 2 sphere.

Originally it was our intention to focus the paper exclusively on codimension 2 spheres in codimension 1 manifolds, but this intention faded as it was seen that the results developed here would apply automatically to spheres of other codimensions and to cell-like subsets as well. For further extensions in the spirit of stated results concerning cell-like subsets, one could make use of [6] to obtain results concerning compacta contained in codimension 1 manifolds and having the shape of polyhedra, but we do not discuss such theorems, leaving them instead to the reader.

The last section of this paper provides specific examples to which the results mentioned above apply.

For definitions of other terms used throughout, the reader is referred to [5,7,18,19].

2. Complements that have the homotopy type of S^1

In this section we mention two properties implying that the complement of a codimension 2 sphere in S^n has the homotopy type of S^1. Since the proofs involve classical covering-space techniques and are similar in nature, we shall simply sketch a proof for one of them.

Lemma 2.1. *Suppose that $S^{n-1} \subset S^n$ is an $(n-1)$-sphere having complementary domains W_1 and W_2 and that $K \subset S^{n-1}$ is an $(n-2)$-sphere such that the inclusion homomorphism $(\pi_1(\mathrm{cl}\, W_e - K) \to \pi_1(S^n - K))$ is trivial ($e = 1,2$). Then $S^n - K$ has the homotopy type of S^1.*

Lemma 2.2. *Suppose that C is an $(n-1)$-cell in S^n such that $\pi_1(S^n - C) = 1$. Then $S^n - \partial C$ has the homotopy type of S^1.*

Proof of Lemma 2.1. Let $\widetilde{X}$ denote the universal covering space of $S^n - K$ and p the projection. By [22, Theorem 3] it suffices to show that $\pi_1(S^n - K) \cong Z$ and $H_i(\widetilde{X}) = 0$ $(i \geqslant 2)$.

Let D_1 and D_2 denote the components of $S^{n-1} - K$. (Note that neither cl D_1 nor cl D_2 need be an $(n-1)$-cell.) Choose points $x_1 \in D_1$ and $x_2 \in D_2$. Associated with each point $\tilde{x}_e$ in the fiber $p^{-1}(x_e)$ $(e = 1,2)$ there are unique liftings $\widetilde{W}_1$ and $\widetilde{W}_2$ of cl $W_1 - K$ and cl $W_2 - K$, respectively; thus, associated with any lifting $\widetilde{W}_e$ of cl $W_e - K$, there are precisely two liftings $\widetilde{W}_j^1$ and $\widetilde{W}_j^2$ of cl $W_j - K$ $(j \neq e)$ such that $\widetilde{W}_j^1 \cup \widetilde{W}_e \cup \widetilde{W}_j^2$ is connected. Given any compact subset A of $\widetilde{X}$, we obtain a finite sequence of liftings $\widetilde{W}_1^1, \widetilde{W}_2^1, \widetilde{W}_1^2, \widetilde{W}_2^2, \ldots, \widetilde{W}_1^k, \widetilde{W}_2^k$ such that

$$A \subset \bigcup_{i=1}^{k} (\widetilde{W}_1^i \cup \widetilde{W}_2^i) = \widetilde{X}_A,$$

and the union of any two successive elements in the sequence is connected. The Mayer–Vietoris Sequence of the triad ($S^n - K$, cl $W_1 - K$, cl $W_2 - K$) reveals that

$$0 \approx H_*(\text{cl } W_e - K) \approx H_*(\widetilde{W}_e^i).$$

By repeated application of the Mayer–Vietoris Sequence, we find that $H_*(\widetilde{X}_A) \approx 0$, which means that $H_*(\widetilde{X}) \approx 0$. Finally, by studying the covering translations of $\widetilde{X}$, one can determine that $\pi_1(S^n - K) \approx Z$. □

3. Unsnarled subsets

Let S denote an $(n-1)$-manifold embedded as a closed separating subset of an n-manifold Q, and let W denote a component of $Q - S$. (All manifolds herein are connected and without boundary.) Whenever Q is a PL manifold and $X \subset S$ is a polyhedron embedded as a locally tame subset of Q, then X is unsnarled from W. Moreover, X is unsnarled from W if X is a cellular subset of Q, or if X simply satisfies McMillan's Cellularity Criterion (see Theorem 3.6). The next lemma and its corollaries disclose relationships between being unsnarled from W and satisfying a one-sided cellularity criterion from W.

Lemma 3.1. *Suppose S is an $(n-1)$-manifold embedded as a closed separating subset of an n-manifold Q, W is a component of $Q - S$, X is* *a closed subset of S that is unsnarled from W, U is an open subset of Q containing X, and f is a map of a 2-simplex Δ^2 into* cl W *such that* $f(\partial\Delta^2) \subset \mathrm{cl}\, W - X$. *Then there exists a map g of Δ^2 into* $(U \cup \mathrm{cl}\, W) - X$ *such that*

$$g \mid \partial\Delta^2 \cup f^{-1}(\mathrm{cl}\, W - U) = f \mid \partial\Delta^2 \cup f^{-1}(\mathrm{cl}\, W - U).$$

Proof. By restricting U, if necessary, we can assume $f(\partial\Delta^2) \cap U = \emptyset$. According to [5, Corollary 2C.2.1] or [7, Corollary 3.3], there exists a map h of Δ^2 into cl W such that

$$h \mid \partial\Delta^2 \cup f^{-1}(\mathrm{cl}\, W - U) = f \mid \partial\Delta^2 \cup f^{-1}(\mathrm{cl}\, W - U),$$

$$h^{-1}(U \cap S) \text{ is 0-dimensional.}$$

Now the compact set $h^{-1}(X) \subset h^{-1}(U \cap S)$ can be covered by the interiors of pairwise disjoint 2-cells $D_1, D_2, ..., D_k$ in Int $\Delta^2 \cap h^{-1}(U)$ so small that to each index i there correspond an $x \in X$ and a neighborhood N_x of x such that $h(D_i) \subset N_x$ and each loop in $N_x \cap W$ is contractible in $U - X$. We produce the required map g by extending $h|\Delta^2 - \bigcup \mathrm{Int}\, D_i$ via an appropriate contraction of each loop $h(\partial D_i)$. □

Corollary 3.2. *Let S, Q and W satisfy the hypothesis of Lemma* 3.1. *If X is a cell-like subset of S and X is unsnarled from W, then for each open set U containing X there exists an open set V such that $X \subset V \subset U$ and each loop in* $V \cap (\mathrm{cl}\, W - X)$ *is contractible in $U - X$.*

Proof. Inasmuch as cl W is an ANR and the cell-like property is invariant under embeddings in ANR's [18], there exists an open set V such that $X \subset V \subset U$ and $V \cap \mathrm{cl}\, W$ is contractible in $U \cap \mathrm{cl}\, W$. The desired conclusion is then a consequence of Lemma 3.1. □

Corollary 3.3. *Let S, Q and W satisfy the hypothesis of Lemma* 3.1. *If the closed subset X of S is a simply connected ANR that is unsnarled from W, then for each open set U containing X there exists an open set V such that $X \subset V \subset U$ and each loop in* $V \cap (\mathrm{cl}\, W - X)$ *is null-homotopic in $U - X$. [Moreover, if X is a compact ANR in S such that $\pi_1(X)$ is abelian and X is unsnarled from W, then for each open set U containing X there exists an open set V such that $X \subset V \subset U$ and each loop in* $V \cap (\mathrm{cl}\, W - X)$ *that is null-homologous in V is null-homotopic in $U - X$.]*

Proof. Again we use the fact that cl W is an ANR, and we specify an open set Z containing cl W and a retraction R of Z to cl W. Given an open set U containing X, we can determine another open set V such that $X \subset V \subset R^{-1}(U \cap \text{cl } W)$ and V deformation retracts to X in $R^{-1}(U \cap \text{cl } W)$. Let $F : V \times I \to R^{-1}(U \cap \text{cl } W)$ denote such a deformation retraction. Then RF defines a homotopy between any loop L in $V \cap \text{cl } W$ and a loop L' in X. For the case $\pi_1(X)$ abelian, if L is null-homologous in V, we see that L' is null-homotopic in X because

$$L' = RF_1(L) \subset RF_1(V) \subset X$$

implies that L' is null-homologous in X. Under either hypothesis, therefore, L is null-homotopic in $U \cap \text{cl } W$, and the corollary follows. □

Lemma 3.4. *Suppose S is a simply connected $(n-1)$-manifold embedded as a closed subset of a simply connected n-manifold Q, W is a component of $Q - S$, and X is a closed subset of S that is unsnarled from W. Then the inclusion induced homomorphism $\pi_1(\text{cl } W - X) \to \pi_1(Q - X)$ is trivial.*

Proof. Note that S must separate Q since Q is simply connected. Each of cl W, $Q - W$ and $S = \text{cl } W \cap (Q - W)$ is an ANR; hence Van Kampen's Theorem implies that cl W is simply connected, and Lemma 3.1 implies that $\pi_1(\text{cl } W - X) \to \pi_1(Q - X)$ is the trivial homomorphism. □

Theorem 3.5. *Let S denote a closed $(n-1)$-manifold in S^n with complementary domains W_1 and W_2. A k-sphere $K \subset S$ is globally 1-alg iff K is unsnarled from both W_1 and W_2.*

Proof. Assume that K is globally 1-alg. For each open set $U \supset K$ there exists an open set $V \supset K$ such that any loop in $V - K$ that is null-homologous in $V - K$ is null-homotopic in $U - K$. In view of the classical result that W_e $(e = 1,2)$ is locally 1-connected in the homology sense at each $s \in S$, for every $x \in X$ we can find a neighborhood N_x such that each loop in $N_x \cap W_e$ is null-homologous in $V \cap W_e$. It follows that every loop in $N_x \cap W_e$ is null-homotopic in $U - K$, and K is unsnarled from W_e. [For this part of the proof it would suffice for K to be an arbitrary closed subset.]

For the other implication, we first focus on the case $k = n - 2$. Given an open set $U \supset K$, we find by Corollary 3.3 a neighborhood V of K such that each loop in $V \cap (\text{cl } W_e - K)$ $(e = 1,2)$ (which is null-homologous in V if $k = 1$) is contractible in $U - K$, and by further restricting V, such that $V \cap (S - K)$ consists of two components. We now establish slightly more than the stated definition of globally 1-alg, by showing that each loop J in $V - K$ that is null-homologous in $U - K$ is null-homotopic in $U - K$. Without loss of generality we consider J to be a simple closed curve that intersects S at just a finite number of points and pierces S at each of them. In case J does not intersect S, the construction of V implies that J is contractible in $U - K$. Otherwise, since J is null homologous in $S^n - K$, J contains an arc α such that $\operatorname{Int} \alpha \subset S^n - S$ and $\partial\alpha$ is contained in one component of $V \cap (S - K)$ [5, Addendum to Theorem 4.1]. Connect the two points of $\partial\alpha$ with an arc β in $V \cap (S - K)$. Then $L = \alpha \cup \beta \subset V \cap (\mathrm{cl}\, W_e - \Sigma)$ is contractible in $U - K$, and J is homotopic in $U - K$ to the curve $J' = (J - \alpha) \cup \beta \subset V - K$. By pushing J' towards the appropriate component of $S^n - S$ at points of β, we move J' homotopically in $V - K$ to a curve J'' having fewer points of intersection with S than J does. Since J, J' and J'' are all homotopic in $U - K$, it follows that J'' is null-homologous in $U - K$ and, by induction, that both J'' and J are null-homotopic in $U - K$.

In case $1 \leqslant k < n - 2$, given $U \supset K$, we see, using the definition of unsnarled, that $x \in K$ has a neighborhood N_x such that any loop in $N_x \cap (\mathrm{cl}\, W_e - K)$ ($e = 1,2$) is contractible in $U - K$, and we can require that $N_x \cap (S - K)$ is connected. Then Van Kampen's Theorem yields that any loop in $N_x - K$ is null-homotopic in $U - K$, from which one can deduce that K is globally 1-alg (see [5, Proposition 1.5]).

The case $k = n - 1$ follows easily since $K = S \simeq S^{n-1}$. □

A similar argument applies to the following theorem.

Theorem 3.6. *Let S denote a closed $(n - 1)$-manifold in S^n having complementary domains W_1 and W_2. A cell-like set X in S satisfies the cellularity criterion iff X is unsnarled from both W_1 and W_2.*

Theorem 3.7. *Suppose that S is an $(n - 1)$-sphere in S^n ($n \neq 4$) with complementary domains W_1 and W_2. A k-sphere K in S is unsnarled from both W_1 and W_2 iff K is weakly flat.*

Proof. According to Theorem 3.5, K is unsnarled from both W_1 and W_2 iff K is globally 1-alg. Whenever $k = n - 2$ and K is unsnarled, it follows from Lemma 3.4 and Lemma 2.1 that $S^n - K$ has the homotopy type of S^1. Thus, for $n \geqslant 5$ the Weak Flatness Criterion gives the desired result, and for $n = 3$ the theorem follows from [8, Theorem 3] or [19, Theorem 4]. □

Theorem 3.8. *Let K denote the boundary of a cellular $(n-1)$-cell C in S^n $(n \geqslant 5)$. Then K is weakly flat.*

Proof. Since $S^n - K$ has the homotopy type of S^1 by Lemma 2.2, it suffices to show that K is globally 1-alg.

Let U be an open set containing K. In this paragraph we mimic the argument in [13, Theorem 3.1] to prove that there exists an open set V containing K such that any loop in $V - C$ is null-homotopic in $U - K$. Choose an $(n-1)$-cell C^* in Int C such that $C - \text{Int } C^* \subset U$, and let $K^* = \partial C^*$. Determine a neighborhood N of K^* such that N deformation retracts to K^* in $U - K$. Now choose neighborhoods N' of $C - C^*$ and N'' of Int C^* such that $N' \subset U$ and $N' \cap N'' = \emptyset$. Since C is cellular, there exists a neighborhood V' of C such that each loop in $V' - C$ is null-homotopic in $(N \cup N' \cup N'') - C$. Define V as $N' \cap V'$. Then each map f of $\partial\Delta^2$ into $V - C$ extends to a map f of Δ^2 into $(N \cup N' \cup N'') - C$. Because $N' \cap N'' = \emptyset$, there exists a disk with holes H such that $\partial\Delta^2 \subset H \subset \Delta^2$, $f(H) \subset N' \cup N$, and $f(\partial H - \partial\Delta^2) \subset N$. All that remains is to define f on each component D of $\text{cl}(\Delta^2 - H)$ so that $f(D) \subset U - K$. This is possible because $\pi_1(N) \to \pi_1(U - K)$ is trivial.

An argument parallel to that given for Theorem 3.5 establishes that any loop in $V - K$ that is null-homologous in $U - K$ is null-homotopic in $U - K$. Consequently, K is globally 1-alg. □

Remark. This argument showing K to be globally 1-alg succeeds only for $n \geqslant 4$ because it requires that the $(n-2)$-sphere K^* be simply connected. In dimension 3, Theorem 3.8 is false; for any 2-cell C on the Fox–Artin sphere S_{FA} (the boundary of the 3-cell obtained by fattening [15, Example 1.2]) such that the wild point of S_{FA} belongs to ∂C, C is cellular but ∂C is not unsnarled from the wild side of S, and consequently ∂C cannot be weakly flat. In view of [13, Theorem 4.1], Theorem 3.8 and [2], the only remaining question, excluding the case $n = 4$, about weak flatness of boundaries of cellular cells pertains to 2-cells in S^n $(n \geqslant 5)$.

Question 1. *Is the boundary of each cellular 2-cell in S^n weakly flat?*

Theorem 3.9. *Each k-sphere K $(k \geqslant 2)$ in the boundary of a cellular m-cell C in S^n $(n \geqslant 5)$ is weakly flat.*

Proof. For the case $m = n$ we could argue directly that K is unsnarled from both components of $S^n - \partial C$, but we shall present an alternate proof independent of the choice of m.

Prescribe a homeomorphism h from an m-simplex Δ^m onto C. Let B denote the $(k+1)$-cell in Δ^m that is the cone from the barycenter b of Δ^m over $h^{-1}(K)$. We show that $h(B)$ is a cellular subset of S^n.

Any PL triangulation T of $\partial\Delta^m$ induces a triangulation T^* of Δ^m via joining to the barycenter b. For any subcomplex L of T containing $h^{-1}(K)$ (as a point-set rather than as a subcomplex), $b*L$ contains B. According to [20, Theorem 1], $h(b*L)$ is cellular, for $b*T$ collapses to $b*L$. Since L can be chosen so that $h(b*L)$ is contained in a preassigned neighborhood of $h(B)$, $h(B)$ itself is cellular. □

As a consequence of Theorem 3.8 or of Theorem 4.1 in [13], $K = h(\partial B)$ is weakly flat. Obviously an affirmative answer to Question 1 would tell us that each k-sphere in the boundary of a cellular m-cell is weakly flat.

With the techniques employed in proving Theorem 3.9 one also obtains the following analogue for cell-like sets.

Theorem 3.10. *If X is a cell-like set in the boundary of a cellular m-cell in S^n, then X is cellular.*

Essential to these arguments is the hypothesis that the codimension 1 manifold be collared from one side. Generally, a weakly flat $(n-1)$-sphere S in S^n can contain an $(n-2)$-sphere that is not weakly flat in S^n and a cell-like subset that is not cellular in S^n (see [11]). However, in the next section we shall see weak flatness in S^{n+1} of all codimension 2 spheres contained in the suspension of S.

4. Unsnarled subsets of factored manifolds

Lemma 4.1. *Suppose S is an $(n-1)$-manifold embedded as a closed separating subset of an n-manifold Q, and W is a component of $Q - S$. Then every closed $(n-2)$-dimensional subset X of S is unsnarled from W if and only if every closed $(n+k-2)$-dimensional subset Y of $S \times E^k$ is unsnarled from $W \times E^k$.*

Proof. It suffices to consider only the case $k = 1$. Assuming each $X \subset S$ is unsnarled from W, we show that $S \times E^1$ is unsnarled at $y = (s, t) \in Y$.

where $s \in S$ and $t \in E^1$. Let U denote an open subset of $Q \times E^1$ containing Y.

We obtain, as in the proof of [9, Theorem 8], a homeomorphism h of $Q \times E^1$ onto itself such that

$$\begin{aligned}
&h|Q \times E^1 - U = \text{identity},\\
&h(S \times E^1) = S \times E^1,\\
&h(y) = y,\\
&h(Y) \cap (S \times t) \text{ is nowhere dense, hence } (n-2)\text{-dimensional.}
\end{aligned}$$

To outline this, for each point $(s,t) \in (S \times t) \cap U$ and neighborhood $N \times J \subset U$ of (s,t), where J is an interval in E^1, there exists a point $(s',t') \in (N \times J) \cap ((S \times E^1) - Y)$. Changing only E^1-coordinates, we can construct a homeomorphism g of $Q \times E^1$ onto itself leaving points outside $N \times J$ fixed and satisfying $g((s',t')) = (s',t)$. With a sequence of such homeomorphisms calculated to converge to a homeomorphism h, we determine h so that $h((S \times E^1) - Y)$ contains a dense subset of $S \times t$.

Define U_t as $U \cap (Q \times t)$ and Y_t as $h(Y) \cap (S \times t)$. By hypothesis there exists a neighborhood N_s of s in Q such that $N_s \times t \subset U_t$ and any loop in $N_s \times t \cap (\mathrm{cl}(W \times t) - Y_t)$ is null-homotopic in $U_t - Y_t$.

Choose an interval J containing t so that $N_s \times J \subset U = h(U)$, and let f denote a map of $\partial\Delta^2$ into $(N_s \times J) \cap (\mathrm{cl}(W \times E^1) - Y)$. Then f is homotopic in $(N_s \times J) \cap ((\mathrm{cl}\, W \times E^1) - Y)$ to a map f' of $\partial\Delta^2$ into $(N_s \times J) \cap (W \times E^1)$, and f' is homotopic in $(N_s \times J) \cap (W \times E^1)$, via a homotopy changing only second coordinates, to a map f'' of $\partial\Delta^2$ into $(N_s \cap W) \times t$. It follows from properties of N_s that f'' is null-homotopic in $U_t - Y_t$. This means that the original map f is null-homotopic in $U - h(Y)$; equivalently, each map g of $\partial\Delta^2$ into $h^{-1}((N_s \times J) \cap (\mathrm{cl}(W \times E^1) - Y))$ is null-homotopic in $h^{-1}(U - h(Y)) = U - Y$.

To prove the other implication, for any closed $(n-2)$-dimensional subset X of S, the hypothesis that $Y = X \times E^1$ is unsnarled from $W \times E^1$ leads directly to the conclusion that X is unsnarled from W. □

Lemma 4.2. *Suppose S is an $(n-1)$-sphere in S^n and W is a component of $S^n - S$ such that every closed $(n-2)$-dimensional subset X of S is unsnarled from W. Then in $S^{n+k} = \Sigma^k(S^n)$, the k-fold suspension of S^n, every closed $(n+k-2)$-dimensional subset Y of $S' = \Sigma^k(S)$ is unsnarled from $W' = \Sigma^k(W) - S'$.*

Proof. Again it suffices to consider only $k = 1$. Let p_1 and p_{-1} denote the suspension points, and regard $S^{n+1} - \{p_1, p_{-1}\}$ as $S^n \times E^1$ in the natural fashion for which $W' = W \times E^1$ and $S' - \{p_1,p_{-1}\} = S \times E^1$. For any closed subset Y of S', $Y' = Y - \{p_1,p_{-1}\}$ is a closed subset of $S \times E^1$, and according to Lemma 4.1, Y is unsnarled from W' at each point of Y'. We assume $p_1 \in Y$ and prove that Y is unsnarled from W at p_1.

Let U be an open subset of S^{n+1} containing Y. Since S' is an n-sphere and since we may as well assume $n \geq 3$, there exists a small neighborhood N_{p_1} of p_1 such that each map f of $\partial\Delta^2$ in $N_{p_1} \cap W'$ extends to a map F of Δ^2 into $U \cap (\mathrm{cl}\, W' - \{p_1,p_{-1}\})$. As already mentioned, Y is unsnarled from W' at each point of $F(\Delta^2) \cap Y$, and Lemma 3.1 provides a map g of Δ^2 into $U - Y$ such that $g|\partial\Delta^2 = F|\partial\Delta^2 = f$. Thus Y is unsnarled at p_1. □

Theorem 4.3. *Suppose S is an $(n-1)$-sphere in S^n such that every $(n-2)$-dimensional compactum in S is unsnarled from both components of $S^n - S$. Then in $S^{n+k} = \Sigma^k(S^n)$ $(n+k \geq 5)$ every m-sphere $(m \leq n+k-2)$ contained in $\Sigma^k(S) = S'$ is weakly flat and every m-dimensional cell-like subset of S' is cellular.*

Proof. See Lemma 4.2 and Theorems 3.6 and 3.7. □

Unlike Lemma 4.1, Lemma 4.2 does not set forth necessary and sufficient conditions that $(n+k-2)$-dimensional subsets of a suspension sphere be unsnarled from W', and the following result discloses that the given condition is not necessary.

Lemma 4.4. *Suppose S is an $(n-1)$-sphere in S^n $(n \geq 3)$ and W is a component of $S^n - S$ such that $\pi_1(W) \approx 1$. Then each closed subset Y of $S' = \Sigma(S)$ is unsnarled from $W' = \Sigma(W) - S'$.*

Proof. For notational convenience we regard S^{n+1} as a quotient space of $S^n \times [-1,1]$ and assign coordinates (x,t), where $x \in S^n$ and $t \in [-1,1]$, to points of S^{n+1} in the usual way. In particular, we continue to denote the suspension points as p_1 and p_{-1}. By the proof of Lemma 4.2 it suffices to show that Y is unsnarled from W' at each point of $Y - \{p_1,p_{-1}\}$. Let (z,t) be an arbitrary point of Y $(-1 < t < 1)$, and let U be an arbitrary neighborhood of Y.

Case 1: $z \times [t,1] \subset Y$. Then there exists $t_0 \in (t,1)$ such that $W' \cap (S^n \times [t_0,1]) \subset U$, and there exist a neighborhood N_z of z in S^n and $\delta > 0$ such that $N_z \times [t-\delta,1] \subset U$. It follows that any loop in $W' \cap (N_z \times (t-\delta,1))$ is homotopic in $W' \cap U$, under a homotopy changing only second coordinates, to a loop in $W' \cap (N_z \times \{t_0\})$ which is null-homotopic in $W \times \{t_0\} \subset W' \cap U$.

Case 2: $z \times [t,1] \not\subset Y$. Then there exists $t_0 \in (t,1)$ such that $(z,t_0) \notin Y$, and $z \times [t,t_0] \subset U$. Again there exist a neighborhood N_z^* of z in S^n and $\delta > 0$ such that $N_z^* \times [t-\delta,t_0] \subset U$ and $N_z^* \times t_0 \cap Y = \emptyset$. It follows that z has another neighborhood N_z in S^n such that each loop in $N_z \cap W$ is contractible in $N_z^* \cap \operatorname{cl} W$. Each loop in $W' \cap (N_z \times (t-\delta,t_0))$ is homotopic in $U \cap W'$, as before, to a loop in $W' \cap (N_z \times \{t_0\})$ which in turn is contractible in $(N_z^* \times \{t_0\}) \cap \operatorname{cl} W' \subset (U - Y) \cap \operatorname{cl} W'$. □

Theorem 4.5. *Suppose S is an $(n-1)$-sphere in S^n ($n \geq 4$) such that for each component W_e of $S^n - S$, $\pi_1(W_e) \approx 1$ ($e = 1,2$). Then each sphere in $\Sigma(S)$ is weakly flat in $\Sigma(S^n)$ and each cell-like subset of $\Sigma(S)$ is cellular.*

Proof. See Theorems 3.6 and 3.7. □

Notice that in Lemma 4.4 a property stronger than that required in the definition of *unsnarled* is satisfied: each point $y \in Y$ has a neighborhood N_y for which each map of $\partial\Delta^2$ into $N_y \cap W'$ extends to a map of Δ^2 into $(U - Y) \cap \operatorname{cl} W'$. Furthermore, the argument for the second case in Lemma 4.4 applies to give the following two theorems.

Theorem 4.6. *Suppose S is an $(n-1)$-manifold embedded as a closed separating subset of an n-manifold Q. Then every closed subset Y of $S \times E^1$ for which there exists $t \in E^1$ such that $Y \subset S \times (-\infty,t]$ is unsnarled from both components of $(Q - S) \times E^1$. In particular, every compact subset of $S \times E^1$ is unsnarled from both components of $(Q - S) \times E^1$.*

A *closed $(n-1)$-string* X in E^n is the image of a closed embedding of E^{n-1} into E^n. A closed $(n-1)$-string $X \subset E^n$ is said to be *factored* if there is a closed $(n-2)$-string $Y \subset E^{n-1}$ such that $(E^{n-1} \times E^1, Y \times E^1) \approx (E^n, X)$.

Corollary 4.7. *Every sphere in a factored $(n-1)$-string $X \subset E^n$ ($n \geq 5$) is weakly flat and every cell-like set in X is cellular.*

Theorem 4.8. *Let S denote an $(n-1)$-sphere in S^n. Then each closed subset Y of $S' = \Sigma(S)$ containing at most one of the suspension points is unsnarled from both components of $S^{n+1} - S'$.*

Corollary 4.9. *Let S denote an $(n-1)$-sphere in S^n $(n \geqslant 4)$. Then every sphere in $S' = \Sigma(S)$ containing at most one of the suspension points is weakly flat.*

Theorem 4.3 and Corollary 4.9 illustrate the sharpness of the following question.

Question 2. *If each $(n-2)$-sphere contained in the $(n-1)$-sphere $S \subset S^n$ is weakly flat, is each $(n-1)$-sphere in $\Sigma(S) \subset \Sigma(S^n)$ weakly flat?*

Question 3. *Does there exist an $(n-1)$-sphere S in S^n $(n \geqslant 5)$ such that each $(n-2)$-sphere in S is weakly flat but some k-sphere $(1 < k < n-2)$ fails to be weakly flat?*

5. Examples

In Section 3 it was mentioned that the Fox–Artin sphere S_{FA} in S^3 contains simple closed curves that are not weakly flat. Since S_{FA} bounds a cellular 3-cell, the k-fold suspension $\Sigma^k(S_{\mathrm{FA}})$ $(k \geqslant 2)$ bounds a cellular $(k+3)$-cell in S^{k+3}. Theorem 3.9 (or Theorem 4.5) implies that any $(k+1)$-sphere in $\Sigma^k(S_{\mathrm{FA}})$ is weakly flat. As a result, we see that the hypothesis of Question 2 is not a necessary one.

In [10], examples are constructed of cellular n-cells C in S^n $(n \geqslant 5)$ such that ∂C is locally flat modulo a Cantor set that is tame relative to S^n. Rather obviously, such cells cannot be obtained by suspending, and, consequently, the results in Section 4 cannot be applied. Nevertheless, according to Theorem 3.9, each m-sphere $(m \leqslant n-2)$ in ∂C is weakly flat. (In fact, one can prove that each m-sphere M $(m \leqslant n-3)$ in ∂C is flat by showing $S^n - M$ to be 1–LC at points of M.)

Let S_{B} and S_{G} denote the 2-spheres in S^3 described by Bing [1] and Gillman [16], respectively, and let S'_{B} and S'_{G} denote their k-fold $(k \geqslant 2)$ suspensions in S^{k+3}. Bryant [3] established that each 2-complex in S'_{B} or S'_{G} is tame relative to S^{k+3}. Daverman [9] extended Bryant's work to show that any k-complex in S^{**}, the k-fold suspension of an arbitrary $(n-1)$-sphere S^* in S^n, is tame relative to $\Sigma^k(S^n) = S^{n+k}$; hence each m-sphere $(m \leqslant k)$ in S'_{B} or S'_{G} is flat. Now we learn from Corollary 4.3 that each $(k+1)$-sphere in S'_{B} or S'_{G} is weakly flat (for S'_{G} this conclusion can be derived from Theorem 3.9 or Corollary 4.5 as well). One suspects, moreover, that each $(k+1)$-sphere K in S'_{B} or S'_{G} must be flat because K must be locally homotopically unknotted; in particular, if K bounds a $(k+2)$-cell in S'_B or S'_G, one can prove that K is flat.

To obtain an everywhere wild $(n-1)$-sphere S' in S^n in which every $(n-2)$-sphere is weakly flat and some $(n-2)$-sphere is wild, consider a 2-sphere S in S^3 having complementary domains W'_{FA} and W'_B, the closures of which are homeomorphic to the (non-3-cell) crumpled cubes bounded by the Fox–Artin sphere [15] and Bing's sphere [1], respectively. The existence of such a sphere S is guaranteed by [12, Theorem 1] or [14, Theorem 6]. Again a wild 1-sphere J can be found in S by choosing J to contain the points at which S is not collared from W'_{FA}. Then $\Sigma^k(J)$ is a wild $(k+1)$-sphere in $\Sigma^k(S) = S' \subset S^{k+3}$ (see [21, Section 2.6]). To see that each $(k+1)$-sphere M in S' is weakly flat, note first that by Lemma 4.4, M is unsnarled from $\Sigma^k(W'_{FA}) - S'$. In order to prove that M is also unsnarled from the other complementary domain, one must exercise some additional care. Let W'_B denote the component of $S - S_B$ corresponding to W'_B, and let X denote a nowhere dense closed subset of S_B. Then X is unsnarled from W_B in the following sense: for each open set $U \supset X$ and $x \in X$ there exists a neighborhood N_x of x such that any loop in $N_x \cap W_B$ is null-homotopic in $(U - X) \cap \mathrm{cl}\, W_B$. (This follows because, according to Theorem 2.1 and the discussion in [4, Section 1], W_B is 1-ULC in cl $W_B - X$.) Clearly then each nowhere dense closed subset X' of S is unsnarled from W'_B. Lemma 4.2 implies that M is unsnarled in $\Sigma^k(W'_B) - S'$, and Theorem 3.7 then implies that M is weakly flat whenever $k \geqslant 2$.

A great variety of $(n-1)$-spheres in S^n for which all subspheres are weakly flat can be generated in the same way by sewing two crumpled cubes and suspending.

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